The ideal structure of semigroups of linear transformations with upper bounds on their nullity or defect

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Abstract

Suppose V is a vector space with $\dim V = p \geq q \geq \aleph_0$, and let $T(V)$ denote the semigroup (under composition) of all linear transformations of V. For each $\alpha \in T(V)$, let $\ker\alpha$ and $\operatorname{ran}\alpha$ denote the 'kernel' and the 'range' of α, and write $n(\alpha) = \dim\ker\alpha$ and $d(\alpha) = \operatorname{codim}\operatorname{ran}\alpha$. In this paper, we study the semigroups $AM(p,q) = \{\alpha \in T(V) : n(\alpha) < q\}$ and $AE(p,q) = \{\alpha \in T(V) : d(\alpha) < q\}$. First, we determine whether they belong to the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide. Then, for each semigroup, we describe its maximal regular subsemigroup, and we characterise its Green's relations and (two-sided) ideals. As a precursor to further work in this area, we also determine all the maximal right simple subsemigroups of $AM(p,q)$.

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1. Introduction

By definition, every two-sided ideal of a semigroup is one-sided, and several authors have studied semigroups with the converse property: namely, every one-sided ideal is two-sided (that is, so-called *duo semigroups*: see [1] and the references therein). Likewise, it is worth studying semigroups with the **BQ**–property: namely, every bi-ideal is a quasi-ideal. This idea first arose in [4] and it has been considered for various transformation semigroups (see [6] for a brief survey). Indeed, the notions of 'bi-ideal' and 'quasi-ideal' date from over 30 years ago, and the significance of the latter was documented in [10]. In this paper, we consider the **BQ**–property and the ideal structure of certain *linear* transformation semigroups. However, to further explain the background to our work, we need some notation.

Let X be an infinite set with cardinal p and let q be a cardinal such that $\aleph_0 \leq q \leq p$. Let $T(X)$ denote the semigroup under composition of all (total) transformations from X to X. If $\alpha \in T(X)$, we write $\operatorname{ran}\alpha$ for the *range* of α and define the *rank* of α to be $r(\alpha) = |\operatorname{ran}\alpha|$. We also write

$$\begin{array}{ll} D(\alpha) = X \setminus \operatorname{ran}\alpha, & d(\alpha) = |D(\alpha)|, \\ C(\alpha) = \bigcup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}, & c(\alpha) = |C(\alpha)|, \end{array}$$

and refer to these cardinal numbers as the *defect* and the *collapse* of α, respectively.

A transformation $\alpha \in T(X)$ is said to be *almost one-to-one* if $c(\alpha)$ is finite. By an *almost onto transformation of* X we mean $\alpha \in T(X)$ such that $d(\alpha)$ is finite. In [5] Theorems 2.1 and 2.3, Kemprasit showed that $AM(X)$, the semigroup of all almost one-to-one transformations of X, and $AE(X)$, the semigroup of all almost onto transformations of X, do not belong to **BQ**, the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide (here, the notation 'M' signifies 'mono', and 'E' denotes 'epi').

Here, we examine related semigroups defined as follows. Let V be a vector space over a field F with dimension $p \geq \aleph_0$. Let $T(V)$ denote the semigroup (under composition) of all linear transformations from V into itself. Also, let $M(V)$ denote the subsemigroup of $T(V)$ consisting of all one-to-one linear transformations, and let $E(V)$ denote the subsemigroup of $T(V)$ consisting of all onto linear transformations. If $\alpha \in T(V)$, we write $\ker\alpha$ and $\operatorname{ran}\alpha$ for the *kernel* and the *range* of α, and put

$$n(\alpha) = \dim\ker\alpha,\ \ r(\alpha) = \dim\operatorname{ran}\alpha,\ \ d(\alpha) = \operatorname{codim}\operatorname{ran}\alpha.$$

As usual, these are called the *nullity, rank* and *defect* of α, respectively. For cardinals $q \leq p$, we write

$$\begin{array}{l} AM(p,q) = \{\alpha \in T(V) : n(\alpha) < q\}, \quad \text{and} \\ AE(p,q) = \{\alpha \in T(V) : d(\alpha) < q\}. \end{array}$$

Clearly, $M(V) \subseteq AM(p,q)$ and $E(V) \subseteq AE(p,q)$. Because of Example 1 below, we will be interested only in the case that q is infinite. Namnak and Kemprasit showed in [8] Theorems 2.2 and 2.3 that $AM(p,\aleph_0)$ and $AE(p,\aleph_0)$ do not belong to **BQ**. In section 2, we generalise these results: we show that $AM(p,q)$ and $AE(p,q)$ are subsemigroups of $T(V)$; and we also show that they do not belong to **BQ**. For each of the two semigroups, we characterise its regular elements; and using this, we determine its unique maximal regular subsemigroup. In section 3, we characterise the Green's relations and ideals in $AM(p,q)$ and $AE(p,q)$ and in section 4, we describe all the maximal right simple subsemigroups of $AM(p,q)$. In passing, we observe that Kemprasit and Namnak did not study Green's relations and ideals for any of the semigroups which they considered.

2. Basic properties

In what follows, $Y = A \dot{\cup} B$ means Y is a *disjoint* union of A and B, and we write id_Y for the identity transformation on Y.

As an abbreviation, we write $\{e_i\}$ to denote a subset $\{e_i : i \in I\}$ of V, taking as understood that the subscript i belongs to some (unmentioned) index set I. The subspace A of V generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$, and then $\dim A = |I|$.

We adopt the convention introduced in [9]. That is, often it is necessary to define some $\alpha \in T(V)$ by first choosing a basis $\{e_i\}$ for V and some $\{a_i\} \subseteq V$, and then letting $e_i\alpha = a_i$ for each i and extending this action by linearity to the whole of V. To abbreviate matters, we simply say, given $\{e_i\}$ and $\{a_i\}$ within context, that $\alpha \in T(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

Often our argument starts by choosing a basis for $\ker\alpha$ and expanding it to one for a subspace containing $\ker\alpha$: provided no confusion will arise, we use this expression even if α is one-to-one (in which case, $\ker\alpha = \{0\}$ and so it has basis the empty set).

For every $\alpha, \beta \in T(V)$, we have $n(\alpha) \leq n(\alpha\beta)$ and $d(\beta) \leq d(\alpha\beta)$, since $\ker\alpha \subseteq \ker(\alpha\beta)$ and $\mathrm{ran}(\alpha\beta) \subseteq \mathrm{ran}\,\beta$. The fact that the sets $AM(p,q)$ and $AE(p,q)$ are semigroups follows from parts (a) and (b), respectively, of the following result, and our assumption that q is infinite. In effect, this result was proved by Namnak and Kemprasit in [8] pp. 217-218, but we include a brief proof for completeness.

Lemma 1. If $\alpha, \beta \in T(V)$ then

(a) $n(\alpha) \leq n(\alpha\beta) \leq n(\alpha) + n(\beta)$, and

(b) $d(\beta) \leq d(\alpha\beta) \leq d(\alpha) + d(\beta)$.

Proof. Let $\alpha, \beta \in T(V)$ and recall that $(\ker(\alpha\beta))\alpha = \ker\beta \cap \mathrm{ran}\,\alpha$. If $\ker(\alpha\beta) = \ker\alpha \oplus \langle e_j \rangle$ then $(\ker(\alpha\beta))\alpha = \langle e_j\alpha \rangle \subseteq \ker\beta$, so $|J| \leq n(\beta)$ and hence $n(\alpha\beta) = n(\alpha) + |J| \leq n(\alpha) + n(\beta)$. Now suppose $\mathrm{ran}\,\beta = \mathrm{ran}(\alpha\beta) \oplus \langle e_i \rangle$. Then $d(\alpha\beta) = d(\beta) + |I|$, where $|I| = \dim(\mathrm{ran}\,\beta / \mathrm{ran}(\alpha\beta))$. Clearly if $V = (\mathrm{ran}\,\alpha + \ker\beta) \oplus U$, then $d(\alpha) \geq \dim U$ and $\mathrm{ran}\,\beta = \mathrm{ran}(\alpha\beta) \oplus U\beta$ (for, if $w = v\alpha\beta = u\beta$ then $v\alpha - u \in \ker\beta$, so $u \in \mathrm{ran}\,\alpha + \ker\beta$ and this implies $u = 0$, so $w = 0$). Hence $\dim(\mathrm{ran}\,\beta / \mathrm{ran}(\alpha\beta)) = \dim(U\beta) \leq \dim U \leq \dim(V/\mathrm{ran}\,\alpha) = d(\alpha)$, and the result follows. □

Example 1. We note that $AM(p,q)$ and $AE(p,q)$ are semigroups only when q is infinite (or 1). For, suppose q is finite, $q \neq 1$, and let $\{e_i\} \dot{\cup} \{u_1, u_2, \ldots, u_q\}$ be a basis for V, with $|I| = p$. Now define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} u_1 & u_2 & \dots & u_q & e_i \\ 0 & u_2 & \dots & u_q & e_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} u_1 & u_2 & \dots & u_q & e_i \\ u_1 & 0 & \dots & 0 & e_i \end{pmatrix}.$$

Clearly, $n(\alpha) = d(\alpha) = 1$ and $n(\beta) = d(\beta) = q - 1$, and so $\alpha, \beta \in AM(p,q) \cap AE(p,q)$. It is easy to see that $\ker(\alpha\beta) = \langle u_1, u_2, \dots, u_q \rangle$ and $V = \operatorname{ran}(\alpha\beta) \oplus \langle u_1, u_2, \dots, u_q \rangle$. Therefore, $n(\alpha\beta) = d(\alpha\beta) = q$ and hence $\alpha\beta \notin AM(p,q) \cup AE(p,q)$.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. A subsemigroup B of S is a *bi-ideal* of S if $BSB \subseteq B$. Note that every right and every left ideal of S is a quasi-ideal, and every quasi-ideal Q of a semigroup S is a bi-ideal of S since $QSQ \subseteq SQ \cap QS$. Given a non-empty subset X of S, the quasi-ideal and the bi-ideal generated by X will be denoted respectively by $(X)_Q$ and $(X)_B$. If $X = \{x_1, x_2, \dots, x_n\}$ then we write $(x_1, x_2, \dots, x_n)_Q$ and $(x_1, x_2, \dots, x_n)_B$ instead of $(\{x_1, x_2, \dots, x_n\})_Q$ and $(\{x_1, x_2, \dots, x_n\})_B$, respectively. By [2] Vol. 1, pp. 84-85, Exercises 15 and 17, if X is a non-empty subset of a semigroup S, then

$$\begin{aligned} &(X)_Q = S^1X \cap XS^1 = (SX \cap XS) \cup X, \quad \text{and} \\ &(X)_B = (XS^1X) \cup X = XSX \cup X \cup X^2. \end{aligned}$$

It is known that regular semigroups, right [left] simple semigroups and right [left] 0-simple semigroups are in the class **BQ** of all semigroups whose sets of bi-ideals and quasi-ideals coincide (see [8] Propositions 1.2 and 1.3 for references to these results). On the other hand, by [8] Corollary 1.5, if $(x)_B \neq (x)_Q$ for some element x of a semigroup S, then $S \notin \mathbf{BQ}$.

We now decide whether $AM(p,q)$ belongs to **BQ**. For this, we follow the argument for [8] Theorem 2.2, although the latter concerned only the case $q = \aleph_0$.

Theorem 1. For any infinite cardinals $p \geq q$, the semigroup $AM(p,q)$ does not belong to **BQ**.

Proof. Suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{f_i\} \,\dot\cup\, \{f_j\}$ with $|J| = q$. Now write $\{f_j\} = \{a_j\} \,\dot\cup\, \{b_k\}$ with $|K| < q$ and $\{a_j\} = \{g_j\} \,\dot\cup\, \{h_j\}$. Put $\{h_j\} \,\dot\cup\, \{b_k\} = \{c_j\}$ and define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} f_i & f_j \\ f_i & g_j \end{pmatrix}, \qquad \beta = \begin{pmatrix} f_i & f_j \\ f_i & a_j \end{pmatrix}.$$

Since $n(\alpha) = 0 = n(\beta)$, we have $\alpha, \beta \in AM(p,q)$. Now define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} f_i & g_j & h_j & b_k \\ f_i & a_j\alpha & h_j & b_k \end{pmatrix}.$$

Since $\{a_j\alpha\} \subseteq \{f_j\alpha\} = \{g_j\}$, it follows that γ is one-to-one and so $\gamma \in AM(p,q)$. Clearly, $\beta\alpha = \alpha\gamma$ and hence $\beta\alpha \in AM(p,q)\alpha \cap \alpha AM(p,q) = (\alpha)_Q$ (the intersection contains α since $AM(p,q)$ contains id_V).

Suppose $\beta\alpha \in (\alpha)_B$. Then, $\beta\alpha \in \alpha AM(p,q)\alpha \cup \{\alpha\}$ (again, note that $AM(p,q)$ contains id_V, so the first set in this union contains α^2). If $\beta\alpha = \alpha$ then, since α

is one-to-one, $\beta = \mathrm{id}_V$, a contradiction. Thus, there exists $\lambda \in AM(p,q)$ such that $\beta\alpha = \alpha\lambda\alpha$. Since α is one-to-one, it follows that $\beta = \alpha\lambda$. Hence, $\langle f_i, a_j\rangle = \operatorname{ran}\beta = \operatorname{ran}(\alpha\lambda) = (\operatorname{ran}\alpha)\lambda = \langle f_i, g_j\rangle\lambda$ and so $V = \langle f_i, a_j, b_k\rangle = \langle f_i, g_j\rangle\lambda + \langle b_k\rangle$. For each j, $c_j\lambda \in V$, and so there exist $u_j \in \langle f_i, g_j\rangle$ and $v_j \in \langle b_k\rangle$ such that $c_j\lambda = u_j\lambda + v_j$. Then, $(c_j - u_j)\lambda = v_j \in \langle b_k\rangle$. Since $\{c_j\} \dot\cup \{f_i\} \dot\cup \{g_j\}$ is linearly independent, it follows that $\{c_j - u_j\}$ is also linearly independent and $c_r - u_r \neq c_s - u_s$ if $r \neq s$. Let $C = \langle c_j - u_j\rangle$. Then, $\dim C = q$ and $\operatorname{ran}(\lambda|C) \subseteq \langle b_k\rangle$. Hence, $\dim(\operatorname{ran}(\lambda|C)) < q$. Since $q = \dim C = \dim(\ker(\lambda|C)) + \dim(\operatorname{ran}(\lambda|C))$ by the Rank-Nullity Theorem, it follows that $\dim(\ker(\lambda|C)) = q$. But $\ker(\lambda|C) \subseteq \ker\lambda$ and so $n(\lambda) \geq n(\lambda|C) = q$, which contradicts the fact that $\lambda \in AM(p,q)$. Therefore, $\beta\alpha \notin (\alpha)_B$ and so $(\alpha)_Q \neq (\alpha)_B$. By [8] Corollary 1.5, $AM(p,q) \notin \mathbf{BQ}$. □

From a remark before Theorem 1, it follows that the semigroup $AM(p,q)$ is neither regular nor right simple nor left simple, for any infinite cardinals p,q such that $p \geq q$. Hence, it is worth determining all regular elements in $AM(p,q)$.

Theorem 2. Let $\alpha \in AM(p,q)$. Then, α is regular if and only if $\alpha \in AE(p,q)$. Consequently, $AM(p,q) \cap AE(p,q)$ is the largest regular subsemigroup of $AM(p,q)$.

Proof. Suppose $\alpha \in AE(p,q)$. Let $\{e_j\}$ be a basis for $\ker\alpha$ and expand it to a basis $\{e_j\} \dot\cup \{e_i\}$ for V. Now write $e_i\alpha = a_i$ for each i. Since $\{a_i\}$ is a basis for $\operatorname{ran}\alpha$, it can be expanded to a basis for V, say $\{a_i\} \dot\cup \{a_k\}$. Define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} a_i & a_k \\ e_i & 0 \end{pmatrix}.$$

Clearly, $n(\beta) = d(\alpha) < q$ and $d(\beta) = n(\alpha) < q$, and hence $\beta \in AM(p,q) \cap AE(p,q)$. Also, $\alpha = \alpha\beta\alpha$ and so α is regular in $AM(p,q)$. Conversely, suppose $\alpha = \alpha\beta\alpha$ for some $\beta \in AM(p,q)$. Then $\beta\alpha$ is an idempotent in $T(V)$, so $V = \ker(\beta\alpha) \oplus \operatorname{ran}(\beta\alpha)$ and, since $AM(p,q)$ is closed, it follows that $q > n(\beta\alpha) = d(\beta\alpha) \geq d(\alpha)$. Therefore, $\alpha \in AE(p,q)$ as required.

Finally, given a regular subsemigroup S of $AM(p,q)$, we know it is contained in $AE(p,q)$, and so $S \subseteq AM(p,q) \cap AE(p,q)$. Thus, the latter is the largest regular subsemigroup of $AM(p,q)$. □

Similar results hold for the semigroup $AE(p,q)$, as we now proceed to show. In the proof of our next theorem, we use an argument similar to the one used in [8] Theorem 2.3, but ours is complicated by the possibility that $q > \aleph_0$.

Theorem 3. For any infinite cardinals $p \geq q$, the semigroup $AE(p,q)$ does not belong to $\mathbf{BQ}$.

Proof. Suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{f_i\} \dot\cup \{h\}$. Now write $\{f_i\} = \{a_i\} \dot\cup \{b_i\}$ and define $\alpha, \beta \in T(V)$ by

$$\alpha = \begin{pmatrix} a_i & b_i & h \\ f_i & 0 & h \end{pmatrix}, \qquad \beta = \begin{pmatrix} a_i & b_i & h \\ a_i & b_i & 0 \end{pmatrix}.$$

Since $d(\alpha) = 0$ and $d(\beta) = \dim\langle h\rangle = 1 < q$, we have $\alpha, \beta \in AE(p,q)$. Also, $\alpha \neq \beta\alpha = \alpha\beta$ and so $\alpha\beta \in AE(p,q)\alpha \cap \alpha AE(p,q) = (\alpha)_Q$ (note that the intersection contains α, since $AE(p,q)$ contains id_V).

Now suppose $\alpha\beta \in (\alpha)_B = \alpha AE(p,q)\alpha \cup \{\alpha\}$ (again, note that $AE(p,q)$ contains id_V, and so the first set in this union contains α^2). Then, since $\alpha\beta \neq \alpha$, we know $\alpha\beta = \alpha\lambda\alpha$ for some $\lambda \in AE(p,q)$ and the surjectivity of α implies $\beta = \lambda\alpha$. Thus, $(h\lambda)\alpha = h(\lambda\alpha) = h\beta = 0$ and so $h\lambda \in \ker\alpha$. Hence, there exist a natural number n and scalars $x_1, \ldots, x_n$ such that

$$h\lambda = \sum_{r=1}^{n} x_r b_{i_r}. \tag{1}$$

Put $\{b_i\} \setminus \{b_{i_1}, \ldots, b_{i_n}\} = \{c_i\}$. We assert that $\{c_i + \operatorname{ran}\lambda\}$ is a linearly independent subset of $V/\operatorname{ran}\lambda$. Suppose $\sum y_i(c_i + \operatorname{ran}\lambda) = \operatorname{ran}\lambda$ for some scalars y_i. Then, $\sum y_i c_i \in \operatorname{ran}\lambda$ and so there exists some $u \in V$ such that $\sum y_i c_i = u\lambda$. Since $V = \langle a_i\rangle \oplus \langle b_i\rangle \oplus \langle h\rangle$, there exist scalars r_i and s, and a vector $v \in \langle a_i\rangle$, such that $u = v + \sum r_i b_i + sh$. Hence,

$$\sum y_i c_i = v\lambda + \sum r_i(b_i\lambda) + s(h\lambda). \tag{2}$$

Thus,

$$\sum y_i(c_i\alpha) = v(\lambda\alpha) + \sum r_i(b_i\lambda\alpha) + s(h\lambda\alpha).$$

Since $\ker\alpha = \langle b_i\rangle$, $\lambda\alpha = \beta$ and $\ker\beta = \langle h\rangle$, it follows that $0 = v\beta + \sum r_i(b_i\beta)$. That is, $v + \sum r_i b_i \in \ker\beta$ and, by our choice of bases, this implies $v = 0$ and $r_i = 0$ for each i. Thus, we can rewrite (2):

$$\sum y_i c_i = s(h\lambda).$$

From (1),

$$\sum y_i c_i = \sum_{r=1}^{n} (sx_r) b_{i_r}.$$

Since $\{c_i\} \dot\cup \{b_{i_1}, \ldots, b_{i_n}\}$ is linearly independent, it follows that $y_i = 0$ for each i. Hence, $\{c_i + \operatorname{ran}\lambda\}$ is linearly independent, and so $q > d(\lambda) = \dim(V/\operatorname{ran}\lambda) = p$, a contradiction. Therefore, $\alpha\beta \notin (\alpha)_B$ and so $(\alpha)_B \neq (\alpha)_Q$. Hence, by [8] Corollary 1.5, $AE(p,q) \notin \mathbf{BQ}$. □

From the previous Theorem, it follows that $AE(p,q)$ is neither regular nor right simple or left simple, for any infinite cardinals p and q such that $p \geq q$. In the next result, we determine all regular elements in $AE(p,q)$.

Theorem 4. Let $\alpha \in AE(p,q)$. Then, α is regular if and only if $\alpha \in AM(p,q)$. Consequently, $AM(p,q) \cap AE(p,q)$ is the largest regular subsemigroup of $AE(p,q)$.

Proof. By Theorem 2, if $\alpha \in AM(p,q)$ then $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$ for some $\beta \in AM(p,q)$, and hence $\beta \in AE(p,q)$ (by Theorem 2 again). That is, every $\alpha \in AM(p,q) \cap AE(p,q)$ is a regular element of $AE(p,q)$. Conversely, suppose $\alpha \in AE(p,q)$ and $\alpha = \alpha\beta\alpha$ for some $\beta \in AE(p,q)$. Then $\alpha\beta$ is an idempotent in $T(V)$, and hence $V = \ker(\alpha\beta) \oplus \operatorname{ran}(\alpha\beta)$ and, since $AE(p,q)$ is closed, it follows that $q > d(\alpha\beta) = n(\alpha\beta) \geq n(\alpha)$. Therefore, $\alpha \in AM(p,q)$ as required. Finally, as in the last paragraph of the proof of Theorem 2, $AM(p,q) \cap AE(p,q)$ is the largest regular subsemigroup of $AE(p,q)$. □

3. Green's relations and ideals

Green's relations on $T(V)$ are well-known: if $\alpha, \beta \in T(V)$, then $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$; $\alpha \mathcal{R} \beta$ if and only if $\ker\alpha = \ker\beta$; and $\mathcal{D} = \mathcal{J}$ [2] Vol. 1, Exercise 2.2.6. Moreover, by Hall's Theorem ([3], Proposition II.4.5), any regular subsemigroup of $T(V)$ inherits characterisations of its Green's relations from those on $T(V)$. From section 2, we know $AM(p,q)$ and $AE(p,q)$ are not regular, so it is surprising that, nonetheless, the $\mathcal{L}$–relation on $AM(p,q)$ and the $\mathcal{R}$–relation on $AE(p,q)$ can be described just like the corresponding ones on $T(V)$, and moreover $\mathcal{D} = \mathcal{J}$ for both semigroups. On the other hand, their ideal structure differs markedly from that of $T(V)$, as we eventually show in this section.

First, we characterise the $\mathcal{L}$ relation on $AM(p,q)$ and the $\mathcal{R}$ relation on $AE(p,q)$.

Lemma 2. Let $\alpha, \beta \in AM(p,q)$. Then $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

Proof. Suppose $\operatorname{ran}\alpha = \operatorname{ran}\beta$ and let $\{e_j\}$ be a basis for $\ker\beta$. Expand $\{e_j\}$ to a basis $\{e_j\} \dot\cup \{e_i\}$ for V and write $e_i\beta = b_i$ for each i. Then, $\{b_i\}$ is a basis for $\operatorname{ran}\beta = \operatorname{ran}\alpha$. For every i, choose $f_i \in b_i\alpha^{-1}$. Clearly, $\{f_i\}$ is linearly independent. Now define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & f_i \end{pmatrix}.$$

Since $\ker\lambda = \ker\beta$, it follows that $\lambda \in AM(p,q)$. Also, $\beta = \lambda\alpha$. Similarly, we conclude that there exists $\mu \in AM(p,q)$ such that $\alpha = \mu\beta$, and so $\alpha \mathcal{L} \beta$. The converse involves a standard argument, so we omit the details. □

Lemma 3. Let $\alpha, \beta \in AE(p,q)$. Then $\alpha \mathcal{R} \beta$ if and only if $\ker\alpha = \ker\beta$.

Proof. Suppose $\ker\alpha = \ker\beta$ and let $\{e_j\}$ be a basis for this subspace. Expand $\{e_j\}$ to a basis $\{e_j\} \dot\cup \{e_i\}$ for V and, for each i, write $e_i\alpha = a_i$ and $e_i\beta = b_i$. Clearly, $\{a_i\}$ and $\{b_i\}$ are bases for $\operatorname{ran}\alpha$ and $\operatorname{ran}\beta$, respectively. Now expand $\{b_i\}$ to a basis for V, say $\{b_i\} \dot\cup \{b_\ell\}$, and define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} b_\ell & b_i \\ 0 & a_i \end{pmatrix}.$$

Since $d(\lambda) = d(\alpha)$, it follows that $\lambda \in AE(p,q)$. Also, $\alpha = \beta\lambda$. Similarly we conclude that there exists $\mu \in AE(p,q)$ such that $\beta = \alpha\mu$. Hence $\alpha \mathcal{R} \beta$. The converse involves a standard argument, so we omit the details. □

We proceed to characterise the $\mathcal{R}$ relation on $AM(p,q)$. For this, we need two preliminary Lemmas.

Lemma 4. If $\alpha, \beta, \lambda \in T(V)$ satisfy $\alpha = \beta\lambda$ then

$$d(\beta) \leq n(\lambda) + \dim(\operatorname{ran}\lambda / \operatorname{ran}\alpha).$$

In fact, if we also have $\ker\alpha = \ker\beta$, then $d(\beta) = n(\lambda) + \dim(\operatorname{ran}\lambda / \operatorname{ran}\alpha)$.

Proof. Since $\alpha = \beta\lambda$ implies $\ker\beta \subseteq \ker\alpha$, we can write $\ker\beta = \langle e_r \rangle$, $\ker\alpha = \langle e_r, e_s \rangle$ and $V = \langle e_r \rangle \oplus \langle e_s \rangle \oplus \langle e_j \rangle$. Write $e_j\alpha = a_j$, $e_s\beta = b_s$ and $e_j\beta = b_j$, and note that $a_j = e_j\alpha = (e_j\beta)\lambda = b_j\lambda$ for each j. In addition, $\{a_j\}$ and $\{b_s, b_j\}$ are bases for $\operatorname{ran}\alpha$

and $\operatorname{ran}\beta$, respectively. Now, if $\sum x_j b_j \in \ker\lambda$ for some scalars x_j, then $\sum x_j a_j = 0$ and so $x_j = 0$ for each j: that is, $\langle b_j\rangle \cap \ker\lambda = \{0\}$. Hence we can write $V = \langle b_j\rangle \oplus \ker\lambda \oplus \langle e_k\rangle$ and we assert that $\operatorname{ran}\lambda = \operatorname{ran}\alpha \oplus \langle e_k\lambda\rangle$. For, if $\sum x_j a_j = \sum y_k(e_k\lambda)$ for some scalars x_j and y_k then $\sum x_j b_j - \sum y_k e_k \in \ker\lambda$ and, by our choice of bases, this implies $x_j = 0 = y_k$ for all j and k. Clearly, $\{e_k\lambda\}$ is linearly independent. Since $\langle b_j\rangle \subseteq \operatorname{ran}\beta$, we have

$$d(\beta) \leq \operatorname{codim}\langle b_j\rangle = n(\lambda) + |K| = n(\lambda) + \dim(\operatorname{ran}\lambda/\operatorname{ran}\alpha).$$

Finally, if we also have $\ker\alpha = \ker\beta$ then, with the previous notation, $\operatorname{ran}\beta = \langle b_j\rangle$ and $V = \langle b_j\rangle \oplus \ker\lambda \oplus \langle e_k\rangle$ and so $d(\beta) = n(\lambda) + |K|$. $\square$

Lemma 5. If $\alpha, \beta \in AM(p,q)$ and $\alpha \mathcal{R} \beta$, then $\alpha \in AE(p,q)$ if and only if $\beta \in AE(p,q)$.

Proof. Suppose the conditions hold and $\alpha \in AE(p,q)$. By Theorem 2, α is a regular element of $AM(p,q)$, and so D_α, the $\mathcal{D}$-class of α in $AM(p,q)$, is regular (by [2] Vol. 1, Theorem 2.11). Now let R_α denote the $\mathcal{R}$-class of α in $AM(p,q)$. Since $\beta \in R_\alpha \subseteq D_\alpha$, this implies β is a regular element of $AM(p,q)$ and so $\beta \in AE(p,q)$ by Theorem 2. Similarly, if $\beta \in AE(p,q)$ then $\alpha \in AE(p,q)$. $\square$

Lemma 6. Let $\alpha \in AM(p,q)$ and denote the $\mathcal{R}$-class of $AM(p,q)$ containing α by R_α. Then,

(a) $\alpha \in AE(p,q)$ implies $R_\alpha = \{\beta \in AM(p,q) : \beta \in AE(p,q) \text{ and } \ker\beta = \ker\alpha\}$;

(b) $\alpha \notin AE(p,q)$ implies $R_\alpha = \{\beta \in AM(p,q) : \ker\beta = \ker\alpha \text{ and } d(\beta) = d(\alpha)\}$.

Proof. First suppose $\alpha \in AE(p,q)$. If $\beta \in AM(p,q)$ is such that $\alpha \mathcal{R} \beta$, then, since $\mathrm{id}_V \in AM(p,q)$, there exist $\lambda, \mu \in AM(p,q)$ such that $\alpha = \beta\lambda$ and $\beta = \alpha\mu$. Therefore $\ker\alpha = \ker\beta$. Also, we know $\beta \in AE(p,q)$, from Lemma 5.

Conversely, suppose $\beta \in AM(p,q) \cap AE(p,q)$ and $\ker\beta = \ker\alpha$. Since $AM(p,q) \cap AE(p,q)$ is a regular subsemigroup of $AE(p,q)$, Hall's Theorem ([3], Proposition II.4.5) implies that the $\mathcal{R}$ relation on $AM(p,q) \cap AE(p,q)$ is the restriction of the $\mathcal{R}$ relation on $AE(p,q)$ to $AM(p,q) \cap AE(p,q)$. In other words, since $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ and $\ker\alpha = \ker\beta$, we deduce from Lemma 3 that $\alpha \mathcal{R} \beta$ in $AM(p,q) \cap AE(p,q)$ and hence $\alpha \mathcal{R} \beta$ in $AM(p,q)$. That is, $\beta \in R_\alpha$ as required, and (a) holds.

Now, suppose $\alpha \notin AE(p,q)$ and $\alpha \mathcal{R} \beta$ in $AM(p,q)$. Then $\beta \notin AE(p,q)$ (by Lemma 5) and $\alpha = \beta\lambda$, $\beta = \alpha\mu$ for some $\lambda, \mu \in AM(p,q)$. As we already know, the latter implies $\ker\alpha = \ker\beta$. Moreover, since $\alpha = \beta\lambda$, $n(\lambda) < q$ and $d(\beta) \geq q$, by Lemma 4 we have $d(\beta) \leq \dim(\operatorname{ran}\lambda/\operatorname{ran}\alpha) \leq \dim(V/\operatorname{ran}\alpha) = d(\alpha)$. Similarly, since $\beta = \alpha\mu$, $n(\mu) < q$ and $d(\alpha) \geq q$, we deduce that $d(\alpha) \leq d(\beta)$ and equality follows.

Conversely, suppose $\beta \in AM(p,q)$ is such that $\ker\beta = \ker\alpha$ and $d(\beta) = d(\alpha)$. Let $\{e_j\}$ be a basis for $\ker\alpha = \ker\beta$, with $|J| = n(\alpha) = n(\beta)$, and expand it to a basis $\{e_j\} \dot\cup \{e_i\}$ for V. Now write $e_i\alpha = a_i$ and $e_i\beta = b_i$ for each i. Then, $\{a_i\}$ is a basis for $\operatorname{ran}\alpha$ and it can be expanded to a basis for V, say $\{a_i\} \dot\cup \{a_k\}$, where $|K| = d(\alpha) \geq q$. Similarly, $\{b_i\}$ is a basis for $\operatorname{ran}\beta$ and we can expand it to a basis $\{b_i\} \dot\cup \{b_k\}$ for V (note that $d(\beta) = d(\alpha) = |K|$). Since $|K| \geq q$, we can write $\{a_k\}$ as $\{u_k\} \dot\cup \{u_r\}$ and $\{b_k\}$ as $\{v_k\} \dot\cup \{v_r\}$, where $|R| < q$. Now define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} b_i & v_k & v_r \\ a_i & u_k & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i & u_k & u_r \\ b_i & v_k & 0 \end{pmatrix}.$$

Since $n(\lambda) = \dim\langle v_r\rangle < q$ and $n(\mu) = \dim\langle u_r\rangle < q$, we have $\lambda, \mu \in AM(p,q)$. Also, $\alpha = \beta\lambda$ and $\beta = \alpha\mu$. Hence, $\alpha \,\mathcal{R}\, \beta$ and (b) holds. □

The next two results are crucial for the characterisation of the $\mathcal{L}$ relation on $AE(p,q)$.

Lemma 7. If $\alpha, \beta, \lambda \in T(V)$ satisfy $\alpha = \lambda\beta$, then

$$n(\beta) \leq d(\lambda) + \dim(\ker\alpha/\ker\lambda).$$

In fact, if $\operatorname{ran}\alpha = \operatorname{ran}\beta$ then $n(\beta) = d(\lambda) + \dim(\ker\alpha/\ker\lambda)$.

Proof. Since $\alpha = \lambda\beta$, we can write $\ker\lambda = \langle e_j\rangle$, $\ker\alpha = \langle e_j\rangle \oplus \langle e_i\rangle$ and $V = \langle e_j\rangle \oplus \langle e_i\rangle \oplus \langle f_k\rangle$. Write $f_k\alpha = a_k$ and $f_k\lambda = u_k$ for each k, and note that $\{a_k\}$ is a basis for $\operatorname{ran}\alpha$. In addition, $a_k = f_k\alpha = u_k\beta$. Clearly, the set $\{e_i\lambda\} \,\dot\cup\, \{u_k\}$ is linearly independent, and hence $\operatorname{ran}\lambda = \langle e_i\lambda\rangle \oplus \langle u_k\rangle$. Moreover, if $(\sum x_k u_k)\beta = 0$ for some scalars x_k, then $\sum x_k(u_k\beta) = 0$, and hence $\sum x_k a_k = 0$ and so $x_k = 0$ for each k. Thus $\ker\beta \cap \langle u_k\rangle = \{0\}$. Therefore,

$$n(\beta) \leq \operatorname{codim}\langle u_k\rangle = d(\lambda) + |I| = d(\lambda) + \dim(\ker\alpha/\ker\lambda).$$

Now suppose $\operatorname{ran}\beta = \operatorname{ran}\alpha = \langle a_k\rangle$. If $v \in V$, there exist scalars y_k such that $v\beta = \sum y_k a_k$ and so $v\beta = (\sum y_k u_k)\beta$. Hence, $v - \sum y_k u_k \in \ker\beta$ and thus $v \in \ker\beta \oplus \langle u_k\rangle$. Therefore, $V = \ker\beta \oplus \langle u_k\rangle$ and, in this case, $n(\beta) = \operatorname{codim}\langle u_k\rangle = d(\lambda) + \dim(\ker\alpha/\ker\lambda)$. □

Lemma 8. If $\alpha, \beta \in AE(p,q)$ and $\alpha \,\mathcal{L}\, \beta$, then $\alpha \in AM(p,q)$ if and only if $\beta \in AM(p,q)$.

Proof. This is identical to the proof of Lemma 5 using $\mathcal{L}$ in place of $\mathcal{R}$ and Theorem 4 in place of Theorem 2. □

Lemma 9. Let $\alpha \in AE(p,q)$ and denote the $\mathcal{L}$-class of $AE(p,q)$ containing α by L_α. Then,

(a) $\alpha \in AM(p,q)$ implies $L_\alpha = \{\beta \in AE(p,q) : \beta \in AM(p,q) \text{ and } \operatorname{ran}\beta = \operatorname{ran}\alpha\}$;

(b) $\alpha \notin AM(p,q)$ implies $L_\alpha = \{\beta \in AE(p,q) : \operatorname{ran}\beta = \operatorname{ran}\alpha \text{ and } n(\beta) = n(\alpha)\}$.

Proof. Let $\beta \in AE(p,q)$ be such that $\alpha \,\mathcal{L}\, \beta$. Then, there exist $\lambda, \mu \in AE(p,q)$ such that $\alpha = \lambda\beta$ and $\beta = \mu\alpha$ (since $\mathrm{id}_V \in AE(p,q)$) and so $\operatorname{ran}\alpha = \operatorname{ran}\beta$. If $\alpha \in AM(p,q)$, then $\beta \in AM(p,q)$ (by Lemma 8). If $\alpha \notin AM(p,q)$, then $\beta \notin AM(p,q)$ (again, by Lemma 8) and so $n(\alpha) \geq q$ and $n(\beta) \geq q$. From Lemma 7, we know that $n(\beta) \leq d(\lambda) + n(\alpha)$ and, similarly, $n(\alpha) \leq d(\mu) + n(\beta)$. Since $d(\lambda) < q \leq n(\alpha)$ and $d(\mu) < q \leq n(\beta)$, it follows that $d(\lambda) + n(\alpha) = n(\alpha)$ and $d(\mu) + n(\beta) = n(\beta)$. Hence, $n(\beta) = n(\alpha)$.

Conversely, suppose $\alpha \in AM(p,q)$, $\beta \in AM(p,q) \cap AE(p,q)$ and $\operatorname{ran}\beta = \operatorname{ran}\alpha$. Then, as in the proof of Lemma 6, Hall's Theorem together with Lemma 2 imply that $\alpha \,\mathcal{L}\, \beta$ in $AM(p,q) \cap AE(p,q)$ and hence $\alpha \,\mathcal{L}\, \beta$ in $AE(p,q)$. That is, $\beta \in L_\alpha$ as required.

On the other hand, suppose $\alpha \notin AM(p,q)$, $\beta \in AE(p,q)$, $\operatorname{ran}\beta = \operatorname{ran}\alpha$ and $n(\beta) = n(\alpha)$. Let $\operatorname{ran}\alpha = \langle e_i\rangle$, and choose $a_i, b_i \in V$ such that $a_i\alpha = e_i$ and $b_i\beta = e_i$ for each i. Clearly, $\{a_i\}$ is linearly independent. Moreover, if $\ker\alpha = \langle a_k\rangle$ then $V = \langle a_i\rangle \oplus \langle a_k\rangle$:

if $u \in V$ then $u\alpha = \sum x_i e_i = (\sum x_i a_i)\alpha$ for some scalars x_i, so $u - \sum x_i a_i \in \ker \alpha$; and clearly $\{a_i\} \cup \{a_k\}$ is linearly independent. Similarly, $V = \langle b_i \rangle \oplus \langle b_k \rangle$ where $\ker \beta = \langle b_k \rangle$ and $|K| = n(\beta) = n(\alpha)$. Now write

$$\{a_k\} = \{u_k\} \mathbin{\dot\cup} \{u_r\}, \quad \{b_k\} = \{v_k\} \mathbin{\dot\cup} \{v_r\},$$

where $|R| < q$, and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} a_i & u_k & u_r \\ b_i & v_k & 0 \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & v_k & v_r \\ a_i & u_k & 0 \end{pmatrix}.$$

Then $d(\lambda) = |R| < q$, so $\lambda \in AE(p,q)$ and likewise $\mu \in AE(p,q)$. Moreover, $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, so $\alpha \,\mathcal{L}\, \beta$ in $AE(p,q)$ as required. $\square$

Next we describe the $\mathcal{D}$ and $\mathcal{J}$ relations on $AM(p,q)$, and the characterisation of its ideals follows from this.

Theorem 5. If $\alpha, \beta \in AM(p,q)$ then $\alpha \,\mathcal{D}\, \beta$ in $AM(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in AE(p,q)$,

(b) $\alpha, \beta \notin AE(p,q)$ and $d(\alpha) = d(\beta)$.

Proof. Suppose $\alpha \,\mathcal{L}\, \gamma \,\mathcal{R}\, \beta$ in $AM(p,q)$. If $\beta \in AE(p,q)$ then $\gamma \in AE(p,q)$ (by Lemma 5): that is, $d(\gamma) < q$ and, since $\operatorname{ran}\alpha = \operatorname{ran}\gamma$, this implies $d(\alpha) < q$. Hence $\alpha \in AE(p,q)$. On the other hand, if $\beta \notin AE(p,q)$ then, by Lemma 6(b), $d(\alpha) = d(\gamma) = d(\beta) \geq q$ and hence $\alpha \notin AE(p,q)$. For the converse, we start by writing

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} f_k & f_i \\ 0 & b_i \end{pmatrix}$$

(this is possible since $\alpha, \beta \in AM(p,q)$ implies $r(\alpha) = r(\beta) = p$). Now define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} f_k & f_i \\ 0 & a_i \end{pmatrix}.$$

If $\alpha, \beta \in AE(p,q)$, then $n(\gamma) = n(\beta) < q$ and $d(\gamma) = d(\alpha) < q$, so $\gamma \in AM(p,q) \cap AE(p,q)$. In fact, $\operatorname{ran}\gamma = \operatorname{ran}\alpha$ and $\ker\gamma = \ker\beta$, so $\alpha \,\mathcal{L}\, \gamma$ and $\gamma \,\mathcal{R}\, \beta$, and hence $\alpha \,\mathcal{D}\, \beta$ in $AM(p,q)$. However, if $\alpha, \beta \notin AE(p,q)$ and $d(\alpha) = d(\beta)$, then $\gamma \in AM(p,q)$ (as before) and $\operatorname{ran}\gamma = \operatorname{ran}\alpha$, so $\alpha \,\mathcal{L}\, \gamma$ by Lemma 2. Also, $\ker\gamma = \ker\beta$ and $d(\gamma) = d(\alpha) = d(\beta)$, so $\gamma \,\mathcal{R}\, \beta$ by Lemma 6(b). In other words, we have shown that $\alpha \,\mathcal{D}\, \beta$ in $AM(p,q)$. $\square$

Corollary 1. $\mathcal{D} = \mathcal{J}$ on $AM(p,q)$.

Proof. We know $\mathcal{D} \subseteq \mathcal{J}$. Therefore, since $\mathcal{D}$ is universal on $AM(p,q) \cap AE(p,q)$ by Theorem 5(a), $\mathcal{J}$ is also. Now suppose $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \mu, \lambda', \mu' \in AM(p,q)$. By Lemma 4, we have

$$d(\beta) \leq d(\lambda\beta) \leq n(\mu) + \dim(\operatorname{ran}\mu / \operatorname{ran}\alpha) \leq n(\mu) + d(\alpha).$$

Hence if $\beta \notin AE(p,q)$ then $q \leq d(\beta) \leq n(\mu) + d(\alpha)$, and $n(\mu) < q$, so $d(\alpha) \geq q$ and thus $\alpha \notin AE(p,q)$. Likewise, using $\beta = \lambda'\alpha\mu'$, we find that $\alpha \notin AE(p,q)$ implies

$\beta \notin AE(p,q)$. That is, if $\alpha \mathrel{\mathcal{J}} \beta$ in $AM(p,q)$ then either $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ or $\alpha, \beta \notin AE(p,q)$. In the latter case, we have $d(\beta) \leq n(\mu) + d(\alpha) = d(\alpha)$ since $n(\mu) < q \leq d(\alpha)$. Similarly, $\beta = \lambda'\alpha\mu'$ implies $d(\alpha) \leq d(\beta)$, and equality follows. Thus, $\alpha \mathrel{\mathcal{D}} \beta$ by Theorem 5(b). Hence, in both cases, $\alpha \mathrel{\mathcal{J}} \beta$ implies $\alpha \mathrel{\mathcal{D}} \beta$. □

Theorem 6. The proper ideals of $AM(p,q)$ are precisely the sets

$$M_\xi = \{\alpha \in AM(p,q) : d(\alpha) \geq \xi\},$$

where $q \leq \xi \leq p$. In fact, each M_ξ is a principal ideal of $AM(p,q)$ generated by an element with defect ξ.

Proof. Let ξ be a cardinal such that $q \leq \xi \leq p$. By Lemma 4, given $\alpha \in M_\xi$ and $\lambda, \mu \in AM(p,q)$, we have

$$\xi \leq d(\alpha) \leq d(\lambda\alpha) \leq n(\mu) + \dim(\operatorname{ran}\mu/\operatorname{ran}(\lambda\alpha\mu)) \leq n(\mu) + d(\lambda\alpha\mu).$$

Since $n(\mu) < q$ and $\xi \geq q$, we see that $d(\lambda\alpha\mu) \geq \xi$. Therefore, $\lambda\alpha\mu \in M_\xi$ and so M_ξ is an ideal of $AM(p,q)$ (note that λ and μ can equal $\mathrm{id}_V \in AM(p,q)$).

Conversely, let I be an ideal of $AM(p,q)$. If there exists $\alpha \in I \cap AE(p,q)$ then $\alpha \in AM(p,q) \cap AE(p,q)$ and, since $\mathrm{id}_V \in AM(p,q) \cap AE(p,q)$, Theorem 5(a) implies $\mathrm{id}_V \mathrel{\mathcal{D}} \alpha$. Consequently, by Corollary 1, we have $\mathrm{id}_V \in J(\alpha)$, the principal ideal of $AM(p,q)$ generated by α, so $\mathrm{id}_V \in I$ and hence $I = AM(p,q)$. Now suppose $I \cap AE(p,q) = \emptyset$ and choose $\gamma \in I$ with minimal defect ξ. Note that $d(\beta) \geq d(\gamma) = \xi$ for every $\beta \in I$ and, clearly, $q \leq \xi \leq p$. Hence,

$$AM(p,q)\gamma AM(p,q) \subseteq I \subseteq M_\xi.$$

Given $\alpha \in M_\xi$, we have $d(\alpha) \geq \xi = d(\gamma)$. In the usual way, write

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} f_k & f_i \\ 0 & b_i \end{pmatrix}$$

(note that this is possible since $\alpha, \gamma \in AM(p,q)$ implies $r(\alpha) = r(\gamma) = p$). Since $\{b_i\}$ is a basis for $\operatorname{ran}\gamma$, it can be expanded to a basis for V, say $\{b_i\} \dot{\cup} \{b_\ell\}$, with $|L| = d(\gamma) = \xi$. Similarly, $\{a_i\}$ is a basis for $\operatorname{ran}\alpha$ and it can be expanded to a basis $\{a_i\} \dot{\cup} \{a_r\} \dot{\cup} \{a_\ell\}$ for V, where $|R| + |L| = d(\alpha)$ (note that $d(\alpha) \geq d(\gamma) = |L|$). Now define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & f_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_\ell \\ a_i & a_\ell \end{pmatrix}.$$

Clearly, $n(\lambda) = n(\alpha) < q$ and $n(\mu) = 0$, and hence $\lambda, \mu \in AM(p,q)$. Also $\alpha = \lambda\gamma\mu$ and, since I is an ideal, $\gamma \in I$ implies $\alpha \in I$. Therefore, $I = M_\xi$ and, in effect, we have shown that I is a principal ideal generated by an element with defect ξ. □

Clearly, the proper ideals of $AM(p,q)$ form a chain under $\subseteq$, with the smallest being M_p and the largest being M_q.

Now we proceed to characterise the $\mathcal{D}$ and $\mathcal{J}$ relations on $AE(p,q)$ and, using this, we describe the ideal structure of $AE(p,q)$.

Theorem 7. If $\alpha, \beta \in AE(p,q)$ then $\alpha\ \mathcal{D}\ \beta$ in $AE(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in AM(p,q)$,

(b) $\alpha, \beta \notin AM(p,q)$ and $n(\alpha) = n(\beta)$.

Proof. Suppose $\alpha\ \mathcal{L}\ \gamma\ \mathcal{R}\ \beta$ in $AE(p,q)$. If $\alpha \in AM(p,q)$ then $\gamma \in AM(p,q)$ (by Lemma 8) and hence $n(\gamma) < q$. Since $\ker\gamma = \ker\beta$, we have $n(\beta) = n(\gamma) < q$ and so $\beta \in AM(p,q)$. Conversely, if $\alpha, \beta \in AM(p,q) \cap AE(p,q)$ then the same argument as that used in the proof of Theorem 5(a) shows that $\alpha\ \mathcal{D}\ \beta$ in $AE(p,q)$.

Now assume $\alpha\ \mathcal{L}\ \gamma\ \mathcal{R}\ \beta$ in $AE(p,q)$ and $\alpha \notin AM(p,q)$. Then, $\ker\beta = \ker\gamma$ and so, by Lemma 9(b), $n(\beta) = n(\gamma) = n(\alpha) \geq q$, so $\beta \notin AM(p,q)$. Conversely, suppose $\alpha, \beta \notin AM(p,q)$ and $n(\alpha) = n(\beta)$ and, in the usual way, write

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} f_j & f_i \\ 0 & b_i \end{pmatrix}$$

(note that this is possible since $d(\alpha) < q$ and $d(\beta) < q$ imply $r(\alpha) = r(\beta) = p$). Now define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} f_j & f_i \\ 0 & a_i \end{pmatrix}.$$

Then, $d(\gamma) = d(\alpha) < q$, so $\gamma \in AE(p,q)$. In fact, $\ker\gamma = \ker\beta$ and so $\gamma\ \mathcal{R}\ \beta$. Also, $\operatorname{ran}\gamma = \operatorname{ran}\alpha$ and $n(\gamma) = n(\beta) = n(\alpha)$. Hence $\alpha\ \mathcal{L}\ \gamma$. In other words, we have shown $\alpha\ \mathcal{D}\ \beta$ in $AE(p,q)$. $\square$

Corollary 2. $\mathcal{D} = \mathcal{J}$ on $AE(p,q)$.

Proof. Since $\mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{D}$ is universal on $AM(p,q) \cap AE(p,q)$, so is $\mathcal{J}$. Now suppose $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \mu, \lambda', \mu' \in AE(p,q)$. By Lemma 7, it follows that

$$n(\beta) \leq n(\beta\mu) \leq d(\lambda) + \dim(\ker\alpha/\ker\lambda) \leq d(\lambda) + n(\alpha).$$

Therefore, if $\beta \notin AM(p,q)$ then $q \leq n(\beta) \leq d(\lambda) + n(\alpha)$, and $d(\lambda) < q$, so $n(\alpha) \geq q$. Hence $\alpha \notin AM(p,q)$. Likewise, using $\beta = \lambda'\alpha\mu'$, we conclude that $\alpha \notin AM(p,q)$ implies $\beta \notin AM(p,q)$. Thus, if $\alpha\ \mathcal{J}\ \beta$ in $AE(p,q)$, then $\alpha \in AM(p,q)$ if and only if $\beta \in AM(p,q)$. Moreover, if $\alpha, \beta \notin AM(p,q)$ then $n(\beta) \leq d(\lambda) + n(\alpha) = n(\alpha)$ and $n(\alpha) \leq d(\lambda') + n(\beta) = n(\beta)$. Hence $n(\alpha) = n(\beta)$ and so $\alpha\ \mathcal{D}\ \beta$ by Theorem 7(b). Thus we have shown that $\mathcal{J} \subseteq \mathcal{D}$ on $AE(p,q)$. $\square$

Theorem 8. The proper ideals of $AE(p,q)$ are precisely the sets

$$E_\xi = \{\alpha \in AE(p,q) : n(\alpha) \geq \xi\},$$

where $q \leq \xi \leq p$. In fact, each E_ξ is a principal ideal of $AE(p,q)$ generated by an element with nullity ξ.

Proof. Let ξ be an infinite cardinal such that $q \leq \xi \leq p$, and suppose $\alpha \in E_\xi$ and $\lambda, \mu \in AE(p,q)$. By Lemma 7, we have

$$\xi \leq n(\alpha) \leq n(\alpha\mu) \leq d(\lambda) + \dim(\ker(\lambda\alpha\mu)/\ker\lambda) \leq d(\lambda) + n(\lambda\alpha\mu).$$

Since $\lambda \in AE(p,q)$, we know $d(\lambda) < q$, and $q \leq \xi$ by supposition. Hence $n(\lambda\alpha\mu) \geq \xi$ and so $\lambda\alpha\mu \in E_\xi$. Therefore, E_ξ is an ideal of $AE(p,q)$, since $\mathrm{id}_V \in AE(p,q)$.

Conversely, let I be an ideal of $AE(p,q)$. If there exists $\alpha \in I \cap AM(p,q)$ then $\alpha \in AE(p,q) \cap AM(p,q)$ and, since $\mathrm{id}_V \in AE(p,q) \cap AM(p,q)$, Theorem 7(a) implies $\mathrm{id}_V \ \mathcal{D}\ \alpha$. Consequently, by Corollary 2, we have $\mathrm{id}_V \in J(\alpha)$, the principal ideal of $AE(p,q)$ generated by α, so $\mathrm{id}_V \in I$ and hence $I = AE(p,q)$. Finally, suppose $I \cap AM(p,q) = \emptyset$ and choose $\epsilon \in I$ with minimal nullity ξ. Then, $q \leq \xi \leq p$ and $n(\beta) \geq n(\epsilon) \geq \xi$ for every $\beta \in I$. Therefore,

$$AE(p,q)\epsilon AE(p,q) \subseteq I \subseteq E_\xi.$$

Let $\alpha \in E_\xi$. Then $n(\alpha) \geq \xi = n(\epsilon)$. Now let $\{f_k\}$ be a basis for $\ker\epsilon$, with $|K| = \xi$, and expand it to a basis for V, say $\{f_k\} \dot\cup \{f_i\}$. For every i, write $f_i\epsilon = b_i$. Clearly, $\{b_i\}$ is a basis for $\operatorname{ran}\epsilon$, and $\epsilon \in AE(p,q)$ implies $|I| = r(\epsilon) = p$. Likewise, let $\{e_j\} \dot\cup \{e_k\}$ be a basis for $\ker\alpha$, with $|J| + |K| = n(\alpha) \geq n(\epsilon) = |K|$, and expand it to a basis $\{e_j\} \dot\cup \{e_k\} \dot\cup \{e_r\}$ for V. For each r, write $e_r\alpha = a_r$. Since $\alpha \in AE(p,q)$ and $\{a_r\}$ is a basis for $\operatorname{ran}\alpha$, we know $r(\alpha) = p$, and hence we can write $\{e_i\}$ and $\{a_i\}$ instead of $\{e_r\}$ and $\{a_r\}$, respectively. Expand $\{b_i\}$ to a basis for V, say $\{b_i\} \dot\cup \{b_\ell\}$, and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_k & e_i \\ 0 & f_k & f_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_\ell \\ a_i & 0 \end{pmatrix}.$$

Clearly, $d(\lambda) = 0$ and $d(\mu) = d(\alpha) < q$, and hence $\lambda, \mu \in AE(p,q)$. Also, $\alpha = \lambda\epsilon\mu$ and so $\alpha \in I$, since I is an ideal of $AE(p,q)$ and $\epsilon \in I$. Therefore, $I = E_\xi$ and, in effect, we have shown that I is a principal ideal generated by an element with nullity ξ. □

It is now easy to see that the proper ideals of $AE(p,q)$ form a chain under $\subseteq$, with the smallest being E_p and the largest being E_q.

4. Maximal right simple subsemigroups

In [7] Theorem 7, the author proved that if $q \leq \xi \leq p$, then the *linear Baer-Levi semigroups*

$$GS(p,\xi) = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) = \xi\},$$

are precisely the maximal right simple subsemigroups of $KN(p,q) = \{\alpha \in T(V) : n(\alpha) = 0,\ d(\alpha) \geq q\}$ when $q < p$. It is not difficult to show that each $GS(p,\xi)$ is a maximal right simple subsemigroup of $AM(p,q)$ (even if $p = q$). In fact, we will determine all maximal right simple subsemigroups of $AM(p,q)$. To do this, we need two preliminary results.

Lemma 10. For each infinite cardinal ξ such that $\xi \leq p$, and for each subspace A of V with $\dim A < q$, the set

$$M(A,\xi) = \{\alpha \in T(V) : \ker\alpha = A,\ \operatorname{ran}\alpha \cap A = \{0\},\ \dim V/(\operatorname{ran}\alpha \oplus A) = \xi\}$$

is a maximal right simple subsemigroup of $AM(p,q)$.

Proof. Clearly, $M(A,\xi) \subseteq AM(p,q)$ and it is non-empty. For example, if $V = \langle a_j\rangle \oplus \langle a_i\rangle$ where $A = \langle a_j\rangle$ and $|I| = p$ (possible since $\dim A < q \leq p$), we can write

$\{a_i\} = \{b_i\} \dot\cup \{b_k\}$ where $|K| = \xi$ and define $\pi \in M(A, \xi)$ by

$$\pi = \begin{pmatrix} a_j & a_i \\ 0 & b_i \end{pmatrix}.$$

Let $\alpha, \beta \in M(A, \xi)$. Then, $(\ker(\alpha\beta))\alpha = \operatorname{ran}\alpha \cap \ker\beta = \{0\}$ and so $\ker(\alpha\beta) \subseteq \ker\alpha$. Since $\ker\alpha \subseteq \ker(\alpha\beta)$ always, it follows that $\ker(\alpha\beta) = A$. Also $\operatorname{ran}(\alpha\beta) \subseteq \operatorname{ran}\beta$ implies $\operatorname{ran}(\alpha\beta) \cap A \subseteq \{0\}$, and equality follows. Now suppose $\{a_j\}$ is a basis for A and expand it to a basis $\{a_j\} \dot\cup \{a_i\}$ for V, with $|I| = \operatorname{codim} A = p$. For each i, write $a_i\alpha = e_i$. Then $\{e_i\}$ is a basis for $\operatorname{ran}\alpha$, and $\operatorname{ran}\alpha \cap A = \{0\}$ implies $V = \langle a_j\rangle \oplus \langle e_i\rangle \oplus \langle e_k\rangle$ for some linearly independent $\{e_k\} \subseteq V$, where $|K| = \dim V/(\operatorname{ran}\alpha \oplus A) = \xi$. Now write $e_i\beta = f_i$ and $e_k\beta = f_k$ for every i and every k, respectively. Since $\ker\beta = A$, we know that $\{f_i\} \dot\cup \{f_k\}$ is a basis for $\operatorname{ran}\beta$, and hence it can be expanded to a basis for V, say $\{f_i\} \dot\cup \{f_k\} \dot\cup \{c_k\} \dot\cup \{a_j\}$ (recall that $\operatorname{ran}\beta \cap A = \{0\}$ and $\dim V/(\operatorname{ran}\beta \oplus A) = \xi = |K|$). Clearly, we have

$$\alpha\beta = \begin{pmatrix} a_j & a_i \\ 0 & f_i \end{pmatrix}.$$

Hence $\dim V/(\operatorname{ran}(\alpha\beta) \oplus A) = \dim\langle f_k, c_k\rangle = \xi + \xi = \xi$ (since ξ is infinite). Therefore, $\alpha\beta \in M(A, \xi)$ and so $M(A, \xi)$ is a subsemigroup of $AM(p, q)$.

Next we show that $M(A, \xi)$ is right simple. To do this, write $a_i\beta = c_i$ for every i. Since $\ker\beta = A$, we know $\{c_i\}$ is a basis for $\operatorname{ran}\beta$, and hence it can be expanded to a basis for V, say $\{c_i\} \dot\cup \{g_k\} \dot\cup \{a_j\}$ (note that $\operatorname{ran}\beta \cap A = \{0\}$ and $\dim V/(\operatorname{ran}\beta \oplus A) = \xi = |K|$). Now write $\{g_k\} = \{u_k\} \dot\cup \{v_k\}$ (possible since $|K| = \xi \geq \aleph_0$) and define λ in $T(V)$ by

$$\lambda = \begin{pmatrix} a_j & e_i & e_k \\ 0 & c_i & u_k \end{pmatrix}.$$

Then, $\ker\lambda = A$, $\operatorname{ran}\lambda \cap A = \{0\}$ and $\dim V/(\operatorname{ran}\lambda \oplus A) = \xi$, so $\lambda \in M(A, \xi)$. Also $\beta = \alpha\lambda$, and we have shown $M(A, \xi)$ is right simple.

Next suppose $M(A, \xi) \subseteq M \subseteq AM(p, q)$ where M is a right simple subsemigroup of $AM(p, q)$. Since $AM(p, q)$ is not right simple (see the remark before Theorem 2), it follows that $M \neq AM(p, q)$. Let $\alpha \in M$ and $\gamma \in M(A, \xi)$. If $\alpha = \gamma$ then $\alpha \in M(A, \xi)$. Suppose $\alpha \neq \gamma$. Both α and γ are elements of M and, since this semigroup is right simple, there exist $\lambda, \mu \in M$ such that $\alpha = \gamma\lambda$ and $\gamma = \alpha\mu$: that is, $\alpha \mathrel{\mathcal{R}} \gamma$ in M, and hence in $AM(p, q)$ also. By Lemma 6 we have $\ker\alpha = \ker\gamma = A$. Now suppose there exists a non-zero $v = u\alpha \in \operatorname{ran}\alpha \cap A$. Then $u \notin A = \ker\alpha$ and so $\ker\gamma \subsetneq A \oplus \langle u\rangle \subseteq \ker(\alpha\gamma)$. From Lemma 6, we deduce that γ and $\alpha\gamma$ are not $\mathcal{R}$-related in $AM(p, q)$, and hence $\alpha\gamma \notin M$ since M is right simple. But this contradicts the fact that M is closed, so $\operatorname{ran}\alpha \cap A = \{0\}$. Next, we claim that $\dim V/(\operatorname{ran}\alpha \oplus A) = \dim V/(\operatorname{ran}\gamma \oplus A)$.

First, since $\lambda, \mu \in M$, an argument similar to the one above shows that $\ker\lambda = A = \ker\mu$ and $\operatorname{ran}\lambda \cap A = \{0\} = \operatorname{ran}\mu \cap A$. Next, we adopt the same notation as in the second paragraph of this proof, albeit for a different α. Now write $a_i\gamma = g_i$ for each i. Then $\{g_i\}$ is a basis for $\operatorname{ran}\gamma$ (since $\ker\gamma = A = \langle a_j\rangle$) and it can be expanded to a basis for V, say $\{g_i\} \dot\cup \{a_j\} \dot\cup \{g_\ell\}$, where $|L| = \dim V/(\operatorname{ran}\gamma \oplus A) = \xi$ since $\gamma \in M(A, \xi)$. Clearly, $e_i = a_i\alpha = g_i\lambda$ for each i and, since $\ker\lambda = A$, we deduce that $\operatorname{ran}\lambda = \langle e_i\rangle \oplus \langle g_\ell\lambda\rangle$. Consequently, since $\operatorname{ran}\alpha = \langle e_i\rangle$ and $\operatorname{ran}\lambda \cap A = \{0\}$, we obtain

$$\dim V/(\operatorname{ran}\alpha \oplus A) = \operatorname{codim}\langle e_i, a_j\rangle = |K| \geq |L|.$$

Likewise, $\gamma = \alpha\mu$ implies $|K| \leq |L|$. Thus, our claim is valid. Hence α belongs to $M(A,\xi)$, and so $M(A,\xi) = M$. Therefore, $M(A,\xi)$ is a maximal right simple subsemigroup of $AM(p,q)$. □

Note that for each cardinal ξ such that $q \leq \xi \leq p$, we have $GS(p,\xi) = M(\{0\},\xi)$, and hence each $GS(p,\xi)$ is a maximal right simple subsemigroup of $AM(p,q)$, as observed before.

Clearly, the general linear group $G(V)$ is a right simple subsemigroup of $AM(p,q)$. In fact, it is maximal under these conditions. For, suppose $G(V) \subseteq M \subseteq AM(p,q)$ for some right simple subsemigroup M of $AM(p,q)$. Then, given $\alpha \in M$ and $\gamma \in G(V)$, we have $\alpha \,\mathcal{R}\, \gamma$ in M and hence also in $AM(p,q)$, so $\ker\alpha = \ker\gamma = \{0\}$ by Lemma 6. In fact, if $\alpha = \gamma\lambda$ and $\gamma = \alpha\mu$ for some $\lambda,\mu \in M$ then, since M is right simple, λ and μ are $\mathcal{R}$–related to $\gamma \in M$ and so $\ker\lambda = \{0\} = \ker\mu$ as before. Therefore, by Lemma 4,

$$d(\gamma) \leq n(\lambda) + d(\alpha) = d(\alpha) \leq n(\mu) + d(\gamma) = d(\gamma).$$

Hence, $d(\alpha) = 0 = n(\alpha)$ and $\alpha \in G(V)$. In fact, the next result gives a class of maximal right simple subsemigroups of $AM(p,q)$ which contains $G(V)$ (with a slight abuse of terminology, we observe that $G(V) = N(B,\zeta)$ precisely when $\zeta = 0$ and $B = \{0\}$).

Lemma 11. For every cardinal $\zeta < q$ and every subspace B of V with dimension ζ, the set

$$N(B,\zeta) = \{\alpha \in T(V) : \ker\alpha = B,\ V = \operatorname{ran}\alpha \oplus B\}$$

is a maximal right simple subsemigroup of $AM(p,q)$.

Proof. Clearly, $N(B,\zeta) \subseteq AM(p,q)$ and it is non-empty. For example, if $V = \langle b_j\rangle \oplus \langle b_i\rangle$ where $B = \langle b_j\rangle$, $|J| = \zeta$ and $|I| = \operatorname{codim} B$, we can define $\alpha \in N(B,\zeta)$ by

$$\alpha = \begin{pmatrix} b_j & b_i \\ 0 & b_i \end{pmatrix}.$$

Let $\alpha,\beta \in N(B,\zeta)$. Then, $\operatorname{ran}\alpha \cap B = \{0\}$ implies $(\ker(\alpha\beta))\alpha = \{0\}$, and hence $\ker(\alpha\beta) \subseteq B$. Since $B = \ker\alpha \subseteq \ker(\alpha\beta)$, we have $\ker(\alpha\beta) = B$. Clearly $\operatorname{ran}(\alpha\beta) \subseteq \operatorname{ran}\beta$. Now, if $v \in V$, then $v = a + b$ for some $a \in \ker\alpha = \ker\beta$ and $b \in \operatorname{ran}\alpha$. Therefore, there exists $u \in V$ such that $b = u\alpha$ and $v\beta = a\beta + b\beta = u(\alpha\beta)$. Hence, $\operatorname{ran}(\alpha\beta) = \operatorname{ran}\beta$ and so $\alpha\beta \in N(B,\zeta)$.

Now suppose $\{b_j\}$ is a basis for B and expand it to a basis $\{b_j\} \dot\cup \{b_i\}$ for V. For each i, write $b_i\alpha = e_i$ and $b_i\beta = f_i$. Since $\{e_i\}$ and $\{f_i\}$ are bases for $\operatorname{ran}\alpha$ and $\operatorname{ran}\beta$, respectively, we have $V = \langle e_i\rangle \oplus \langle b_j\rangle = \langle f_i\rangle \oplus \langle b_j\rangle$. Define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_i & b_j \\ f_i & 0 \end{pmatrix}.$$

Clearly, $\lambda \in N(B,\zeta)$ and $\beta = \alpha\lambda$. In other words, $N(B,\zeta)$ is right simple.

We have just proved that $N(B,\zeta)$ is a right simple subsemigroup of $AM(p,q)$: next we show it is maximal under these conditions. To do this, suppose $N(B,\zeta) \subseteq M \subseteq AM(p,q)$, where M is a right simple subsemigroup of $AM(p,q)$. As before, $M \neq AM(p,q)$ since the latter is not right simple. Now let $\alpha \in M$ and $\gamma \in N(B,\zeta)$. If

$\alpha = \gamma$ then $\alpha \in N(B, \zeta)$. Now suppose $\alpha \neq \gamma$. Clearly, $\alpha, \gamma \in M$ and so $\alpha = \gamma\lambda$ and $\gamma = \alpha\mu$ for some $\lambda, \mu \in M$. Since $d(\gamma) = \zeta < q$, we have $\gamma \in AE(p, q)$, and hence Lemma 6(a) implies $d(\alpha) < q$ and $\ker \alpha = \ker \gamma = B$. As in the proof of Lemma 10, if $\operatorname{ran} \alpha \cap B \neq \{0\}$ then γ and $\alpha\gamma$ are not $\mathcal{R}$-related in $AM(p, q)$, which implies $\alpha\gamma \notin M$, a contradiction. Therefore $\operatorname{ran} \alpha \cap B = \{0\}$. Likewise, by considering $\lambda, \gamma \in M$ and $\mu, \gamma \in M$, we deduce that $\ker \lambda = B = \ker \mu$ and $\operatorname{ran} \lambda \cap B = \{0\} = \operatorname{ran} \mu \cap B$. Suppose $\operatorname{ran} \alpha \oplus B \subsetneq V$ and write $V = \langle e_i \rangle \oplus \langle b_j \rangle \oplus \langle v_s \rangle$, where $\{b_j\}$ is a basis for B, $\{b_j\} \,\dot\cup\, \{b_i\}$ is a basis for V and $e_i = b_i\alpha$ for each i. Since $b_i\gamma = (b_i\alpha)\mu = e_i\mu$ and $V = \langle b_i\gamma \rangle \oplus \langle b_j \rangle$, we have $V = \langle e_i\mu \rangle \oplus \langle b_j \rangle \subsetneq \langle e_i\mu \rangle \oplus \langle v_s\mu \rangle \oplus \langle b_j \rangle \subseteq V$, a contradiction since $S \neq \emptyset$ and $\langle v_s \rangle \cap \ker \mu = \{0\}$. Hence $\operatorname{ran} \alpha \oplus B = V$. Thus, $\alpha \in N(B, \zeta)$ and $M = N(B, \zeta)$. Therefore, $N(B, \zeta)$ is a maximal right simple subsemigroup of $AM(p, q)$. $\square$

Theorem 9. The maximal right simple subsemigroups of $AM(p, q)$ are exactly the sets $M(A, \xi)$, where A is a subspace of V with $\dim A < q$ and ξ is an infinite cardinal such that $\xi \leq p$, and the sets $N(B, \zeta)$, where ζ is a cardinal such that $\zeta < q$ and B is a subspace of V with $\dim B = \zeta$.

Proof. By Lemma 10, each $M(A, \xi)$ is a maximal right simple subsemigroup of $AM(p, q)$; and by Lemma 11, so is each $N(B, \zeta)$. Now suppose M is a maximal right simple subsemigroup of $AM(p, q)$ and let $\alpha \in M$. For every $\beta \in M$, α and β are $\mathcal{R}$-related in $AM(p, q)$, and hence $\ker \alpha = \ker \beta$. Let $A = \ker \alpha$. As in the proof of Lemma 10, if $\operatorname{ran} \beta \cap A \neq \{0\}$ for some $\beta \in M$, then $A \subsetneq \ker(\beta\alpha)$ and so $\beta\alpha \notin M$, a contradiction. Therefore, $\operatorname{ran} \beta \cap A = \{0\}$ for every $\beta \in M$: in particular, we have $d(\beta) \geq \dim A$. Suppose $\beta \neq \alpha$. Since M is right simple, there exist $\lambda, \mu \in M$ such that $\alpha = \beta\lambda$ and $\beta = \alpha\mu$. Since $\lambda, \mu \in M$, we have $\ker \lambda = A = \ker \mu$ and $\operatorname{ran} \lambda \cap A = \{0\} = \operatorname{ran} \mu \cap A$. In fact, using an argument similar to that in the proof of Lemma 10, we can show that $\dim V/(\operatorname{ran} \alpha \oplus A) = \dim V/(\operatorname{ran} \beta \oplus A)$. Let $\xi = \dim V/(\operatorname{ran} \alpha \oplus A)$ and suppose $\xi \geq \aleph_0$. Then, $M \subseteq M(A, \xi)$ and, by the maximality of M, it follows that $M = M(A, \xi)$.

On the other hand, if ξ is finite then it must be 0: that is, we claim that in this case $V = \operatorname{ran} \beta \oplus A$ for every $\beta \in M$. For, suppose $\operatorname{ran} \alpha \oplus A \subsetneq V$ and write, in the usual way,

$$\alpha = \begin{pmatrix} a_j & a_i \\ 0 & e_i \end{pmatrix}.$$

Now expand $\{e_i\}$ to a basis $\{e_i\} \,\dot\cup\, \{a_j\} \,\dot\cup\, \{e_k\}$ for V, with $|K| = \xi < \aleph_0$. Write $e_i\alpha = v_i$ and $e_k\alpha = v_k$ for every i and every k. Since $\{v_i\} \,\dot\cup\, \{v_k\}$ is a basis for $\operatorname{ran} \alpha$, it can be expanded to a basis for V, say $\{v_i\} \,\dot\cup\, \{v_k\} \,\dot\cup\, \{a_j\} \,\dot\cup\, \{f_k\}$ (this is possible since $\operatorname{ran} \alpha \cap A = \{0\}$ and $\dim V/(\operatorname{ran} \alpha \oplus A) = |K|$). Clearly, $\dim V/(\operatorname{ran} \alpha^2 \oplus A) = \dim\langle v_k, f_k \rangle = 2\xi \neq \xi$, a contradiction. Therefore, $V = \operatorname{ran} \alpha \oplus A$ and $V/(\operatorname{ran} \alpha \oplus A) = \{0\}$. Hence, $V/(\operatorname{ran} \beta \oplus A) = \{0\}$ for every $\beta \in M$, and this implies $V = \operatorname{ran} \beta \oplus A$. Thus, $M \subseteq N(A, \dim A)$ and by the maximality of M, we have $M = N(A, \dim A)$, and the result follows. $\square$

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References

1. A. Cherubini and A. Varisco, Semigroups whose proper subsemigroups are duo, Czechoslovak Math. J., 34(109)(4) (1984), 630-644.

2. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Mathematical Surveys, No. 7, Vols. 1 and 2, American Mathematical Society, Providence, RI, 1961 and 1967.

3. J. M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.

4. K. M. Kapp, On bi-ideals and quasi-ideals in semigroups, Publ. Math. Debrecen, 16 (1969), 179-185.

5. Y. Kemprasit, Infinitely many bi-ideals which are not quasi-ideals in some transformation semigroups, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., 44 (2001), 123-128.

6. Y. Kemprasit, Quasi-ideals and bi-ideals in semigroups and rings, pp. 30-46 in *Proceedings of the International Conference on Algebra and its Applications (ICAA 2002) (Bangkok)*, Chulalongkorn Univ., Bangkok, 2002 (available at www.math.sc.chula.ac.th/ICAA2002/main.html).

7. S. Mendes-Gonçalves, Semigroups of injective linear transformations with infinite defect, Comm. Algebra, 34 (1)(2006), 289-302.

8. C. Namnak and Y. Kemprasit, Some semigroups of linear transformations whose sets of bi-ideals and quasi-ideals coincide, pp. 215-224 in *Proceedings of the International Conference on Algebra and its Aplications (ICAA2002) (Bangkok)*, Chulalongkorn Univ., Bangkok, 2002 (available at www.math.sc.chula.ac.th/ICAA2002/main.html).

9. M. A. Reynolds and R. P. Sullivan, Products of idempotent linear transformations, Proc. Royal Soc. Edinburgh, 100A (1985), 123-138.

10. O. Steinfeld, *Quasi-ideals in rings and semigroups*, Disquisitiones Mathematicae Hungaricae, vol. 10, Akadémiai Kiadó, Budapest, 1978.